SYNOPSES

Rank Preservers and Inertia Preservers on Spaces of Hermitian Matrices

by EHUD MOSHE BARUCH[1] and RAPHAEL LOEWY[1]

I. $G(k, 0, n-k)$ Preservers

Let H_n denote the space of all $n \times n$ Hermitian matrices, and let S_n denote the space of all $n \times n$ real symmetric matrices. If $A \in H_n$ has r positive, s negative, and t zero eigenvalues, then we define the inertia of A to be the triple $\mathrm{In}(A) = (r, s, t)$.

Let $G(r, s, t) = \{A \in H_n : \mathrm{In}(A) = (r, s, t)\}$, and let T be a linear operator on H_n. If $T(G(r, s, t)) \subset G(r, s, t)$, then T is called a $G(r, s, t)$ preserver. Let $\rho(A)$ denote the rank of A.

Johnson and Pierce [3, 4] presented the problem of determining all $G(r, s, t)$ preservers. They considered [3] the case where $t = 0$ and T is onto. Helton and Rodman [2] considered the case where $t = 0$ and T is unital.

EXAMPLE 1.

(*) Let

$$T(A) = S^*AS, \tag{1}$$

$$T(A) = S^*A^tS \tag{2}$$

for some nonsingular $S \in M_n(\mathbb{C})$.

(**) Let

$$T(A) = -S^*AS, \tag{1}$$

$$T(A) = -S^*A^tS \tag{2}$$

for some nonsingular $S \in M_n(\mathbb{C})$.

T of the form (*) is a $G(r, s, t)$ preserver for every (r, s, t). T of the form (**) is a $G(r, s, t)$ preserver for the case $r = s$.

Let T be a $G(r, s, t)$ preserver. Johnson and Pierce [4] proved that if T is *invertible* and if (r, s, t) is not one of the triples $\{(n, 0, 0), (0, n, 0), (0, 0, n), (n/2, n/2, 0)\}$, then:

(1) if $r \neq s$, T is of the form (*);
(2) if $r = s$, T is of the form (*) or (**).

It turns out that the same result holds for the inertia class $(n/2, n/2, 0)$. This was proved by Pierce and Rodman [10] for the Hermitian case and by Loewy [9] for the real

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symmetric case. Johnson and Pierce conjectured [3] that:

(A) If $n \geqslant 3$, r and s are positive, and $r \neq s$, then T is of the form (*).
(B) If $n \geqslant 3$, r and s are positive, and $r = s$, then T is of the form (*) or (**).

This conjecture was proved by Johnson and Pierce [4] for the cases $(n-1,1,0)$, $(k+1,k,0)$, $(1,n-1,0)$, $(k,k+1,0)$. Loewy [8] proved (A) for all cases.

The case $(k,0,n-k)$ is different because there exist singular $G(k,0,n-k)$ preservers. A well-known one is

EXAMPLE 2.

$$T(A) = \begin{bmatrix} \text{trace } A & & & & & \\ & \text{trace } A & & & & \\ & & \ddots & & & \\ & & & \text{trace } A & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} k$$

$$= \text{trace } A \cdot I_k \oplus 0_{n-k}.$$

Loewy [7] proved that if $T: H_n \to H_n$ is a linear operator such that $\rho(A) = 1 \Rightarrow \rho(T(A)) = 1$ and $\dim \operatorname{Im}(T) > 1$, then T is of the form (*) or (**). (The analog for S_n also holds.) Thus if T is a $G(1,0,n-1)$ preserver and $\dim \operatorname{Im}(T) > 1$, then T is of the form (*).

EXAMPLE 3. Let K be a positive integer such that $k < n$, and let

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{bmatrix}, \qquad \text{where} \quad A_1 \in H_k.$$

We define

$$T(A) = \begin{bmatrix} A_1 + t_r(A) I_k & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{for all} \quad A \in H_n.$$

It is not difficult to see that T is a singular $G(k,0,n-k)$ preserver and $\dim \operatorname{Im}(T) = k^2$. Generalizing Loewy's result, we have:

THEOREM 1. *Let $0 < k < n$ be a given integer. If $T: H_n \to H_n$ is a $G(k,0,n-k)$ preserver and* $\dim \operatorname{Im}(T) > k^2$, *then T is of the form* (*). [*The analog for S_n is* $\dim \operatorname{Im}(T) > (k+1)k/2$.]

II. Rank-1 Nonincreasing Operators

Let T be a linear operator. If $\rho(A) \leqslant k \Rightarrow \rho(T(A)) \leqslant k$, then T is called rank-k nonincreasing. It is an open problem to determine all rank-k nonincreasing linear operators on $M_n(\mathbb{C})$ and on H_n. But the following theorem shows that the problems are related:

THEOREM 2. *Let $T: H_n \to H_n$ be a rank-k nonincreasing operator. Let $T_1 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be defined by $T_1(A) = T_1(H_1 + iH_2) = T(H_1) + iT(H_2)$, where $A = H_1 + iH_2$ is the standard decomposition of A into Hermitian and skew-Hermitian parts. Then T_1 is a rank-k nonincreasing operator.*

Let F be a field.

Botta [1] proved that if $T: M_n(F) \to M_n(F)$ is rank-1 nonincreasing, then T is of one of the following forms:

(1) $T(A) = UAV$ for some $U, V \in M_n(F)$.
(2) $T(A) = UA^tV$ for some $U, V \in M_n(F)$.
(3) There exist $a_1, \dots, a_n \in F$ and linear functionals $L_1, \dots, L_n$ such that $T(A)_{ij} = a_i L_j(A)$.
(4) There exist $a_1, \dots, a_n \in F$ and linear functionals $L_1, \dots, L_n$ such that $T(A)_{ij} = a_j L_i(A)$.

M. H. Lim [6] proved that if T is a rank-1 nonincreasing linear operator on the space of symmetric matrices over a field F with char $F \neq 2$, then T is of one of the forms:

(1) $T(A) = \lambda S^tAS$ for $\lambda \in F$ and $S \in M_n(F)$,
(2) $T(A) = L(A) \cdot B$ where L is a linear functional and B is a symmetric matrix such that $\rho(B) = 1$.

Using Theorem 2 and Botta's result, we have the following analog:

THEOREM 3. *If T is rank-1 nonincreasing on H_n, then T is of one of the forms:*

(1) $T(A) = \epsilon S^*AS$ *for* $\epsilon = \pm 1$ *and* $S \in M_n(\mathbb{C})$,
(2) $T(A) = \epsilon S^*A^tS$ *for* $\epsilon = \pm 1$ *and* $S \in M_n(\mathbb{C})$,
(3) $T(A) = L(A) \cdot B$ *where L is a linear functional and $B \in H_n$ is of rank* 1.

III. Rank-2 Preservers on H_n

Let k be a positive integer. If $\rho(A) = k \Rightarrow \rho(T(A)) = k$, then T is called a rank-k preserver. It is an open problem to determine all rank-k preservers on H_n.

For rank-1 preservers we had Loewy's result, which can also be proved by Theorem 3. M. H. Lim [5] proved that if $n > 2$ and T is a rank-2 preserver on the space of symmetric matrices over a field F with char $F = 0$, then $T(A) = \lambda S^tAS$ for some $\lambda \in F$ and a nonsingular matrix $S \in M_n(F)$.

For the Hermitian case we have an analog:

THEOREM 4. *If $n > 2$ and $T : H_n \to H_n$ is a rank-2 preserver, then T is of the form* $(*)$ *or* $(**)$.

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On the Proximal Minimization Algorithm with *D*-Functions

by YAIR CENSOR[2] and STAVROS A. ZENIOS[3]

1. *Introduction*

The *proximal minimization algorithm* deals with the optimization problem

$$\begin{array}{ll} \text{minimize} & F(x) \\ \text{subject to} & x \in X, \end{array} \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a given proper convex function and $X \subseteq \mathbb{R}^n$ is a nonempty closed convex subset of the n-dimensional Euclidean space $\mathbb{R}^n$. The approach is based on

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